

Exact Bethe ansatz solution for a quantum field model of interacting scalar fields in quasi-two dimensions

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Abstract

Integrable quantum field models are known to exist mostly in one space-dimension. Exploiting the concept of multi-time in integrable systems and a Lax matrix of higher scaling order, we construct a novel quantum field model in quasi-two dimensions involving interacting fields. The Yang-Baxter integrability is proved for the model by finding a new kind of commutation rule for its basic fields, representing nonstandard scalar fields along the transverse direction. In spite of a close link with the quantum Landau-Lifshitz equation, the present model differs widely from it, in its content and the result obtained. Using further the algebraic Bethe ansatz we solve exactly the eigenvalue problem of this quantum field model for all its higher conserved operators. The idea presented here should instigate the construction of a novel class of integrable field and lattice models and exploration of a new type of underlying algebras.

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1. Introduction and Motivation

Quantization of integrable field models, in spite of their highly nonlinear interactions, and the exact nonperturbative solution of their eigenvalue problem through an algebraic generalization of the Bethe ansatz (ABA) was a real breakthrough in the theory of quantum integrable systems. Universal appeal of this approach was understood, when the same construction was applied successfully to a number of quantum field models of diverse nature, e.g. nonlinear Schrödinger (NLS) equation and derivative NLS equation and NLS field models, sine-Gordon and Liouville models, quantum Landau-Lifshitz equation (LLE) etc., apart from a rich family of quantum lattice models. It is also revealed, that this family of quantum integrable models can be generated from a single ancestor Lax matrix or its q -deformation, exploring the deep reason behind the broader applicability of the method.

Nevertheless, behind the success of this unifying scheme, there seems to be a limiting factor restricting the existing quantum integrable models within the structures defined by the ancestor model Lax operator and confining their construction only to one space-dimension (1d). The Kitaev models, solvable in two space-dimensions, though belong to a different class, seem to be rather exceptions.

Recall, that the well known $(1+1)$ -dimensional NLS equation

$$iq_t = q_{xx} + 2(q^*q)q, \quad (1)$$

with subscripts denoting partial derivatives, extended as an integrable quantum field model involving bosonic scalar field:

$$[q(x, t), q_x^\dagger(x', t)] = -i \delta(x - x'), \quad (2)$$

was solved way back in eighties. A recent proposal on the other hand constructs, going beyond the known ancestor model Lax operator, a new type of integrable 2d quantum NLS field model, using a higher order Lax matrix. At the classical level this quasi- $(2+1)$ dimensional NLS equation may be given by

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$$iq_t + q_{xy} + 2i(q_x^* q - q^* q_x)q = 0, \quad (3)$$

which differs significantly from the standard NLS (1). As shown in KundNPB15, at the quantum level, this quasi-2d NLS model represents a quantum integrable system, where the basic complex scalar field of the model $q(x, y, t)$, satisfies an unusual commutation rule (CR):

$$[q(x, y, t), q_x^\dagger(x', t)] = -2i \delta(y - y'), [q(x, y, t), q^\dagger(x', t)] = 0, \quad (4)$$

along the transverse direction y , widely different from the bosonic CR (2). However, for establishing the universality of this nonstandard approach, one needs at least another example of a quantum integrable model, where the scheme for constructing integrable quasi-2d field models could be applied. Until the date no such proposal for new models, integrable in higher dimensions, exploiting the idea of Lax operator of higher scaling orders KundNPB15 has been offered yet.

Our motivation here therefore, is to construct a significantly new type of quantum field model in quasi-two dimensional space, exploring the concept of multi-time dimension in integrable systems and following the idea of using a higher order Lax operator and at the same time to solve the model exactly by the algebraic Bethe ansatz. The proposed integrable model shows an intimate connection with the quantum LLE qLLE, though there are also wide differences. It is well known that quantum LLE is receiving renewed interest in recent years in connection with the string theory, since the string states are found to be equivalent to its dual gauge theory, represented by the effective LLE model, starting from semiclassical to the exactly solvable quantum level. Therefore, the proposed field model with a close link to the quantum LLE model might also be important from the string theory point of view. Moreover, the underlying algebraic structure of the basic fields involved in our model, guaranteeing the quantum integrability of the system, represents a new fundamental quantum commutator different from all such algebraic relations known for the existing models.

2. Construction of the integrable field model: classical case

Recall that the LLE

$$\mathbf{S}_t = [\mathbf{S}, \mathbf{S}_{xx}], \quad \mathbf{S}^2 = I, \quad (5)$$

involving spin field $\mathbf{S}(x, t) = (S^1, S^2, S^3)$ with the known CR:

$$[S^a(x, t), S^b(x', t)] = \epsilon^{abc} S^c(x, t) \delta(x - x'), \quad (6)$$

is a $(1 + 1)$ dimensional integrable system, both at the classical and the quantum level. Classical LLE is gauge equivalent to the NLS equation NLSgeLLE and similar to the NLS model the quantum LLE satisfies the Yang-Baxter equation with rational R -matrix and is exactly solvable by the Bethe ansatz qLLE. Integrable systems share the exclusive property of association with a Lax operator, which with its several far reaching consequences, may be considered as a strong criterion for the integrability of the model itself. The space-Lax operator

associated with the LLE:

$$U_{lle}(\lambda) = \frac{1}{\lambda} \mathbf{S}, \quad \mathbf{S}^2 = I \quad (7)$$

represents an infinitesimal space-shift operator in the x -direction, associated with the linear Lax equation $\Phi_x = U_{lle}(\lambda)\Phi$ and falls in the standard structure of the rational ancestor model KunduPRL99 with linear dependence on the spectral parameter $\frac{1}{\lambda}$ and on the basic fields. However, for the present model we look for a Lax operator structure with nonlinear dependence on the spectral parameter as well as on the basic fields. Such Lax operators, though known in the literature, mostly have never been used as a quantum Lax operator involved in the quantum integrability and for the construction of quantum model Hamiltonians.

2.1. Lax operator

Using the concept of multi-time dimension and the space-time duality in integrable systems investigated recently KundNPB15, suris, duaNPB16, we use the time-Lax operator of the LLE system

$$U(\lambda) = \frac{2i}{\lambda^2} \mathbf{S} + \frac{i}{2\lambda} (\mathbf{S}\mathbf{S}_x - \mathbf{S}_x\mathbf{S}) \quad (8)$$

and define it as a space-Lax operator along an additional space dimension, defining $U(\lambda)$ as a generator for the shift along the y direction: $\Phi_y = U(\lambda; x, y, t)\Phi$, in a quasi- $(2 + 1)$ dimensional integrable system. Notice, that the space Lax operator (8) is of higher scaling order compared to $U_{lle}(\lambda)$ (7) for the LLE and other known type models. Since x and $\frac{1}{\lambda}$ go as the length L , the scaling dimension (SD) of $U_{lle}(\lambda)$ and other AKNS type models [22] become 1, while for the Lax operator (8) the SD consequently result to 2, with y scaling as L^2 . Note also, that in spite of an intimate connection with the $(1 + 1)$ -dimensional standard LLE model (5), the present model, as we see below, differs widely from it, both at the classical and the quantum level. The current field components $\mathbf{S} = (S^1, S^2, S^3)$ appearing in (8), unlike the spin field CR (6) exhibit unusual characteristics, expressed through novel algebraic relations (9), as a consequence of the integrability of the system. The present model is defined in a $(2 + 1)$ - dimensional space-time (x, y, t) and the related field CR (9), defined at different space points along the y direction and unlike (6) for the LLE model, does not allow the usual constraint $\mathbf{S}^2 = I$.

As predicted in KundNPB15, the models with such Lax matrices of higher scaling order could constitute a new family of integrable systems satisfying the Yang-Baxter equation with fundamentally new algebraic relations. We would find, that the proposed Lax matrix (8) indeed constructs a ultralocal integrable model of the rational class and though bears resemblance with the LLE qLLE, it is defined in higher than 1d space dimensions and exhibits a new commutation relation, significantly different from the well known algebraic relations, like spin algebra, bosonic algebra etc.. In particular, the field components S^a , $a = 1, 2, 3$ in the present quasi- $(2 + 1)$ dimensional nonlinear model may be defined in the classical case, through the Poisson bracket (PB) relations as

$$\{S^a(x, y, t), S_x^b(x, y', t)\} = \frac{i}{2} \delta_{ab} \delta(y - y'), \quad a, b \in [1, 2, 3] \quad (9)$$

Looking closely to relations (9) we can observe several novelties. First, these PB relations for the field \mathbf{S} are defined at space points along the y direction and does not allow \mathbf{S}^2 as a Casimir operator, while those in case of

the known LLE model are valid along the x direction and allows the constraint $S^2 = 1$. Second, the PB relations involve x derivative of the fields, which goes beyond the known algebraic relations like for the spin, boson, fermion etc. (Indeed we are familiar with canonical brackets involving field and its time-derivative only). Third, unlike the spin algebra (6), nontrivial relations in (9) exist only between the same field components, which is same for all individual field components. Importantly, in spite of the appearance of the derivative term in the Lax operator (8) the specific form of (9), as we see below, guarantees the integrability of the system as an ultralocal model. Rewriting the PB relations (9) as

$$\{S^\pm(y, x, t), S_x^\mp(y', x, t)\} = i\delta_{ab}\delta(y - y'), S^\pm = S^1 \pm iS^2 \quad (10)$$

$$\{S^3(y, x, t), S_x^3(y', x, t)\} = i\delta(y - y'), \quad (11)$$

and comparing them with the relation (4) for the nonstandard complex scalar field proposed recently in an integrable quasi-2d NLS model KundNPB15, we notice that S^+ and its conjugate S^- satisfy a relation similar to the nonstandard complex field, with S^3 satisfying also a similar relation, though as a real scalar field. It is interesting to note that, while the basic fields behave like spin fields in the LLE model with respect to the PB (6) stretching along the x -direction and are expressible through bosonic scalar fields through Holstein-Primakov transformation, in the present model the basic fields following the PB relations (9) along the y -direction, behave like scalar fields themselves, revealing their identity as a complex scalar and a real scalar field, with unusual PB relations as (10,11). Another major difference between the present and the known LLE model is, that $S^2 = S^{1^2} + S^{2^2} + S^{3^2} \equiv s^2$ is a Casimir operator for all the components of the spin field in the PB (6) related to the LLE model and therefore one can set the constant $s^2 = 1$, reducing the degrees of freedom of the fields to 2, linking them to a single bosonic field. However, for the present model with PB relation (10,11) the function $\vec{S}^2 = s^2(x, y, t)$ is no longer a Casimir operator resulting no constraints and remains as a derived field, which consequently leaves the independent degrees of the current field to be 3, corresponding to a complex scalar field S^+, S^- together with a real scalar field S^3 .

2.2. Hamiltonian and higher conserved quantities

The Lax operators $U(\lambda)$ in general may be considered as infinitesimal shift generators along different space-time directions, defining the associated linear system, which for (8) takes the form

$$\Phi_y(x, y, t, \lambda) = U(\lambda)\Phi(x, y, t, \lambda), \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (12)$$

where $U(\lambda)$ represents a shift operator along the y direction, which is the relevant direction here, showing a quasi-2d nature of our model. This system with explicit information about the Lax operator is rich enough to generate all higher conserved quantities including the Hamiltonian of the model and to solve the related hierarchy of nonlinear equations through inverse scattering technique (IST). Since we are concerned about the quantum generalization of the model, we do not deal here with the classical solutions of the nonlinear equations through the IST and are interested only in the explicit construction of the conserved quantities C_n , $n = 1, 2, \dots$,

which may be derived from the Lax equation (12) as

$$\ln \phi_1 = i \sum_n C_n \lambda^n, \quad C_n = \int dy \rho_n, \quad (13)$$

where for constructing the densities ρ_n , $n = 1, 2, \dots$ of conserved quantities we may use the matrix elements U_{ij} of the Lax operator:

$$i \sum_n \rho_n \lambda^n = U_{11} + U_{12} \Gamma, \quad \Gamma = \frac{\phi_1}{\phi_2} \quad (14)$$

to derive a Riccati equation of the form

$$\Gamma_y = U_{21} - 2U_{11} \Gamma - U_{12} \Gamma^2, \quad \Gamma = \sum_{n=0} \Gamma_n \lambda^n, \quad (15)$$

where using the expressions of Lax matrix (8) one gets the recurrence equation

$$\Gamma_{ny} = 4S^3 \Gamma_{n+2} + 2(S^- S_x^+ - S^+ S_x^-) \Gamma_{n+1} + 2S^- \sum_{k=0}^{n+2} \Gamma_{n+2-k} \Gamma_k + (S^3 S_x^- - S^- S_x^3) \sum_{l=0}^{n+1} \Gamma_{n+1-l} \Gamma_l \quad (16)$$

for $n > 0$ where $\Gamma_0 = -\frac{S^3+s}{S^-}$, with $s = \sqrt{S^2}$. Recall again that unlike the LLE model, here $s(x, y, t)$ is not a constant but a real field, which makes all the field components S^1, S^2, S^3 to be independent of each other. Solving recurrence relations (16) one gets in the first step

$$\Gamma_1 = \frac{1}{4s} [(S^3 S_x^- - S^- S_x^3) \Gamma_0^2 + (S^- S_x^+ - S^+ S_x^-) \Gamma_0 + (S^3 S_x^+ - S^+ S_x^3)], \quad \Gamma_0 = -\frac{S^3+s}{S^-}, \quad (17)$$

$$\Gamma_2 = \frac{1}{2(2S^3+s)} \left[\frac{1}{2} \Gamma_{0y} - (S^- S_x^+ - S^+ S_x^-) \Gamma_1 + S^- \Gamma_1^2 + (S^3 S_x^- - S^- S_x^3) \Gamma_1 \Gamma_0 \right], \quad (18)$$

etc. Inserting these relations in (13-14) we can derive finally the infinite set of commuting conserved quantities as

$$C_{n-2} = \int dy (2S^- \Gamma_n + (S^3 S_x^- - S^- S_x^3) \Gamma_{n-1}), \quad n \geq 2. \quad (19)$$

Therefore the lower order conserved quantities may be given in the explicit form

$$\begin{aligned} C_{-2} &= \int dy (2S^3 + 2S^- \Gamma_0) = \int dy (S^3 + s), \\ C_{-1} &= \int dy ((S^- S_x^+ - S^+ S_x^-) + (S^3 S_x^- - S^- S_x^3) \Gamma_0 + 2S^- \Gamma_1) \\ C_0 &= \int dy ((S^3 S_x^- - S^- S_x^3) \Gamma_1 + 2S^- \Gamma_2) \\ C_1 &= \int dy ((S^3 S_x^- - S^- S_x^3) \Gamma_2 + 2S^- \Gamma_3) \end{aligned} \quad (20)$$

etc. The Hamiltonian of the model can be defined as

$$H = C_2 = \int dy ((S^3 S_x^- - S^- S_x^3) \Gamma_3 + 2S^- \Gamma_4) \quad (21)$$

where solutions for $\Gamma_2, \Gamma_3, \Gamma_4$ calculated from recurrence relations (16) using (17), (18) are to be inserted, that are straightforward but a bit lengthy, which we omit here. Notice the quasi $(2+1)$ -dimensional nature of

the Hamiltonian, since though (21) with (16, 17, 18) involve both x and y derivatives of the field, the volume integral is taken only along the y direction. The space-asymmetry with the appearance of space derivatives $S_x^a(x, y, t)$ and $S_y^a(x, y, t)$ in an assymmetric way is also explicit.

2.3. Classical Yang-Baxter equation

For proving the complete integrability of a system it is not enough to have all higher conserved quantities C_n , $n = 1, 2, \dots$, but one has to show that they are all independent entries i.e., are in involutions. Therefore, one has to show that the conserved quantities Poisson-commute $\{C_n, C_m\} = 0$ (operator commute for quantum models). For proving this global statement for our model, one may demand a local sufficient relation on the Lax matrix as

$$\begin{aligned} \{U(\lambda, x, y), \otimes U(\mu, x, y')\} &= [r(\lambda - \mu), U(\lambda) \otimes I + I \otimes U(\mu)] \delta(y - y'), \\ r(\lambda - \mu) &= P r_0(\lambda - \mu), \quad P = \frac{1}{2} (I + \sum_{a=1}^3 \sigma^a \otimes \sigma^a), \quad r_0 = \frac{1}{2(\lambda - \mu)}, \end{aligned} \quad (22)$$

which is known as the classical Yang-Baxter equation (CYBE) with the rational $r(\lambda - \mu)$ -matrix along the relevant direction y (trigonometric and elliptic r -matrices are not relevant in the present context). For proving the integrability of the system at a global level together with the sufficient condition (22) one needs also the ultralocality condition

$$\{U(\lambda, y), \otimes U(\mu, y')\} = 0, \text{ at } y \neq y' \quad (23)$$

at different points on the y axis, which follows also from (22).

Note, that CYBE with the same r -matrix as in (22), though along the x direction, is valid also for the known LLE model LLE, which however gives much simpler relations (involving only 2 nontrivial relations) compared to the present case, having 10 nontrivial relations, with few major ones as

$$\begin{aligned} \{U_{11}(\lambda, y), U_{12}(\mu, y')\} &= 2(U_{11}(\mu) - U_{11}(\lambda)) r_0(\lambda - \mu) \delta(y - y'), \\ \{U_{12}(\lambda, y), U_{21}(\mu, y')\} &= (U_{12}(\mu) - U_{12}(\lambda)) r_0(\lambda - \mu) \delta(y - y') \end{aligned} \quad (24)$$

etc. This happens due to much complicated structure of the present Lax operator (8). However, interestingly, all these involved CYBE relations are satisfied simultaneously due to the novel PB relations among the field components of the present model as in (9), or in more elaborate form as

$$\begin{aligned} \{S^3(y), S_x^3(y')\} &= \frac{i}{2} \delta(y - y'), \quad \{S^+(y), S_x^-(y')\} = i \delta(y - y'), \quad \{S^-(y), S_x^+(y')\} = i \delta(y - y') \\ \{S^3(y), S_x^-(y')\} &= \{S^-(y), S_x^-(y')\} = \{S^-(y), S^+(y')\} = 0 \end{aligned} \quad (25)$$

etc.

It is remarkable, that in spite of the presence of a x -derivative term in the Lax matrix (8), it satisfies the necessary ultralocality condition (23) due to the PBs (25). This is because not x but y is the relevant direction here, where the fields commute at space-separated points along y , reflecting the quasi-2d nature of our model

with space-asymmetry. Recall that the fields in the standard LLE the related PBs hold for space points along the x -direction.

Now we switch over to the quantum generalization of our new integrable field model and show that as a quantum field model it passes the criteria of quantum integrability and allows exact Bethe ansatz solution with intriguing properties.

3. Quantum field model and exact solution

For quantum generalization the recommended procedure is to lattice regularize the fields by discretizing the space along the relevant direction $y \rightarrow j$ to obtain $\mathbf{S}(x, y) \rightarrow \mathbf{S}_j(x)$, and express the associated Lax operator of the model (8) in a discretized form: $U^j(\lambda) = I + \Delta U(\lambda, y \rightarrow j)$, with explicit expression for its matrix operator elements as

$$\begin{aligned} U_{11}^j(\lambda) &= I + i\Delta u^j, \quad U_{22}^j(\lambda) = I - i\Delta u^j, \quad u^j = \frac{2}{\lambda^2} S_j^3 + \frac{i}{2\lambda} (S_j^- S_{jx}^+ - S_j^+ S_{jx}^-) \\ U_{12}^j(\lambda) &= i\Delta \left(\frac{2}{\lambda^2} S_{jx}^- + \frac{1}{\lambda} (S_j^- S_{jx}^3 - S_j^3 S_{jx}^-) \right), \quad U_{21}^j(\lambda) = i\Delta \left(\frac{2}{\lambda^2} S_j^+ + \frac{1}{\lambda} (S_j^+ S_{jx}^3 - S_j^3 S_{jx}^+) \right), \end{aligned} \quad (26)$$

where $S_j^a, a = 1, 2, 3$ are now quantum field operators. Note that the lattice regularization is enough to perform here along the y -direction keeping the space variable x to be continuous, since the Lax operator here is defined as a shift operator along y . Nevertheless, it is to be noted, that the $(2+1)$ dimensional field $\mathbf{S}(x, y, t)$ depends on the coordinates x, y, t , where the field \mathbf{S} together with its x -derivatives enter in the Lax operator in a nonlinear form (see (26)), with the lattice regularization needed for $y \rightarrow j$ only. This fact also exhibits a quasi-2d dependence of our field with marked space-asymmetry. In fact the space directions are scaled differently, which is acceptable for nonrelativistic models, as for example in the well known $(2+1)$ -dimensional integrable KP equation KP. the Poisson brackets (25) can be quantized to yield the commutation relations between the components of the field as

$$[S_j^3, S_{kx}^3] = \frac{\alpha}{\Delta} \delta_{jk}, \quad [S_j^-, S_{kx}^+] = -\frac{2\alpha}{\Delta} \delta_{jk}, \quad [S_j^+, S_{kx}^-] = \frac{2\alpha}{\Delta} \delta_{jk}, \quad [S_j^3, S_k^\pm] = [S_j^\pm, S_k^\pm] = 0 \quad (27)$$

etc.

For showing the quantum integrability of the model, the operator elements of the discretized quantum Lax matrix (26) should satisfy certain algebraic commutation relations, which can be given in a compact matrix form by the quantum Yang-Baxter equation (QYBE)

$$R(\lambda - \mu) U^j(\lambda) \otimes U^j(\mu) = U^j(\mu) \otimes U^j(\lambda) R(\lambda - \mu), \quad (28)$$

at each lattice site $j = 1, 2, \dots, N$, together with an *ultralocality* condition

$$[U^j(\lambda) \otimes U^k(\mu)] = 0, \quad j \neq k, \quad (29)$$

Note, that these relations are quantum generalization of the classical equations (22, 23), where the quantum 4×4 R -matrix with nontrivial elements :

$$\begin{aligned} R_{11}^{11} = R_{22}^{22} &\equiv a(\lambda - \mu) = \lambda - \mu + i\alpha, \quad R_{21}^{12} = R_{12}^{21} \\ &\equiv b(\lambda - \mu) = \lambda - \mu, \quad R_{22}^{11} = R_{11}^{22} \equiv c = i\alpha, \end{aligned} \quad (30)$$

is a quantum extension of the the classical r matrix appearing in (22). It is to be noted, that in spite of the presence of a x -derivative term in the quantum Lax operator (26), thanks to the new CRs (27) the necessary ultralocality condition (29) holds. This is because y and not x is the concerned direction here,, where the fields commute at space separated points along $y \rightarrow j$.

If we define a global operator for N -lattice sites as $T(\lambda) = \prod_{j=1}^N U^j(\lambda)$, through the lattice regularized quantum Lax operator $U^j(\lambda)$, which satisfies the QYBE (28) together with (29), then the global monodromy operator $T(\lambda)$ must also satisfy the QYBE [2]

$$R(\lambda - \mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R(\lambda - \mu), \quad T(\lambda) = \begin{pmatrix} A(\lambda), & B(\lambda) \\ B^\dagger(\lambda), & A^\dagger(\lambda) \end{pmatrix}, \quad (31)$$

with the same $R(\lambda - \mu)$ -matrix. This happens due to the coproduct property of the underlying Hopf algebra, which keeps an algebra invariant under its tensor product [24]. This global QYBE (31) serves two important purposes. First, it proves the quantum integrability of the model by showing the mutual commutativity of all conserved operators. Second, it derives the commutation relations between the operator elements of $T(\lambda)$, which are used for the exact algebraic Bethe ansatz solution of the EVP.

In more details: multiplying QYBE (31) from left by R^{-1} , taking the trace from both sides and using the property of cyclic rotation of matrices under the trace, one can show that $\tau(\lambda) = \text{trace } T(\lambda)$ commutes: $[\tau(\lambda), \tau(\mu)] = 0$. This in turn leads to the Liouville integrability condition: $[C_n, C_m] = 0$, $n, m = 1, 2, \dots$, since the conserved set of operators are generated from $\ln \tau(\lambda) = \sum_j C_n \lambda^n$, through expansion in the spectral parameter λ . Following this construction and exploiting the explicit form of the Lax matrix (26), we can derive, in principle, all conserved operators C_n , $n = 1, 2, \dots$ for our model, as given for the classical case in (20).

Therefore, for proving the quantum integrability of the proposed field model, associated with the quantum Lax operator (26), we have to satisfy the QYBE (28) for each matrix elements. However due to the quadratic spectral power dependence of the Lax operator together with its nonlinear dependence on the fields and its more complicated structure, the problem becomes much harder compared to the known quantum LLE model qLLE. However all these relations (as we see below in explicit form) are satisfied due to the new quantum commutation relations (27) for our quantum field, upto order $O(\Delta)$, which however is enough for quantum field models obtained at $\Delta \rightarrow 0$.

Note that comparing with the well known quantum LLE model, where only *two* nontrivial relations appear in the QYBE, the present model brings harder challenges, since in total *ten* nontrivial quantum equations arise in its QYBE, as we discuss below.

3.1. QYBE for the integrable field model

In QYBE (28) with R -matrix (30), for our quantum integrable model we insert the associated discretized quantum Lax matrix U^j as in (26) and look explicitly for the validity of QYBE relations for each of the matrix operator element. We find, that out of total 16 operator relations, except 4 diagonal and 2 extreme off-diagonal terms, all the other 10 relations Q_{kl}^{ij} stand nontrivial and their validity needs to be proved using in particular the operator product relations at the coinciding points:

$$[S_j^3, S_{jx}^3] = \frac{\alpha}{\Delta}, [S_j^-, S_{jx}^+] = -\frac{2\alpha}{\Delta}, [S_j^+, S_{jx}^-] = \frac{2\alpha}{\Delta}, \text{ and } [S_j^3, S_k^\pm] = [S_j^\pm, S_k^+] = 0, \quad (32)$$

at space-separated points, following from the CR (27).

Using the expressions for $a(\lambda - \mu)$, $b(\lambda - \mu)$, c from (30) and CR (32) we may check the validity of

$$Q_{12}^{11} = a U_{11}^j(\lambda) U_{12}^j(\mu) - b U_{12}^j(\mu) U_{11}^j(\lambda) - c U_{11}^j(\mu) U_{12}^j(\lambda) = +O(\Delta^2) = 0, \quad (33)$$

upto order $O(\Delta^2)$. Similarly, one proves the conjugate relations $Q_{21}^{11}, Q_{11}^{21}, Q_{11}^{12}$ and similar relations $Q_{12}^{22}, Q_{21}^{22}, Q_{22}^{12}, Q_{22}^{21}$.

The remaining two relations can also be proved with the use of the same operator product relations (32):

$$Q_{21}^{12} = b [U_{12}^j(\lambda), U_{21}^j(\mu)] + c (U_{22}^j(\lambda) U_{11}^j(\mu) - U_{11}^j(\lambda) U_{22}^j(\mu)) = 0, \quad (34)$$

which holds exactly in all orders of Δ and similarly for the conjugate relation Q_{12}^{21} . This proves thus the validity of all QYBE relations for our quantum quasi-2d NLS field model, associated with the higher Lax operator (26) and algebraic relations (32), obtained at the limit $\Delta \rightarrow 0$.

4. Algebraic Bethe ansatz for the eigenvalue problem

As noted above, the monodromy operator $T(\lambda)$ associated with our quantum Lax operator (26), as guaranteed by QYBE (28) together with the ultralocality condition (29), satisfies also the same QYBE (31) with the rational R -matrix (this is due to the Hopf algebra property chari inherent to this problem). Therefore, we can follow the procedure for the algebraic BA, close to the formulation of the 1d quantum LLE model qLLE. As we have discussed above, $\tau(\lambda) = \text{trace} T(\lambda) = A(\lambda) + A^\dagger(\lambda)$ is linked to the generator of the conserved operators C_n , $n = 1, 2, \dots$, including the Hamiltonian (21). The off-diagonal elements of $T^{12}(\lambda) = B(\lambda)$ and $T^{21}(\lambda) = B^\dagger(\lambda)$, on the other hand, can be considered as generalized *creation* and *annihilation* operators, respectively. For solving the eigenvalue problem (EVP) for all conserved operators: $C_n |M\rangle = c_n^M |M\rangle$, $n = 1, 2, \dots$ simultaneously, we construct exact M-particle Bethe state $|M\rangle = B(\mu_1) B(\mu_2) \cdots B(\mu_M) |0\rangle$, on a pseudo-vacuum $|0\rangle$ with the property $B^\dagger(\mu_a) |0\rangle = 0$, $A(\lambda) |0\rangle = g(\lambda) |0\rangle$, where numerical function $g(\lambda)$ depends on the vacuum expectation value of the Lax operator: $U_0(\lambda) = \langle 0 | U^j(\lambda) | 0 \rangle$ and aim to solve the EVP: $\tau(\lambda) |M\rangle = \Lambda_M(\lambda, \mu_1, \mu_2, \dots, \mu_M) |M\rangle$, with exact eigenvalues $\ln \Lambda_M(\lambda, \{\mu_a\}) = \sum_j c_n^M(\{\mu_a\}) \lambda^n$.

4.1. Exact solution for quasi 2d quantum field model

For obtaining the final result for our quantum field model, on infinite space interval, we have to switch over to the field limit: $\Delta \rightarrow 0$ with total lattice site $N \rightarrow \infty$ and then take the interval $L = N\Delta \rightarrow \infty$, assuming vanishing of the field $S_j^\pm \rightarrow 0$, $S_j^3 \rightarrow 1$, at $j \rightarrow \infty$, compatible with the natural condition of having the vacuum state at space infinities, yielding the asymptotic Lax matrix $U^j(\lambda)|_{j \rightarrow \infty} = U_0(\lambda) = I + \frac{2i}{\lambda^2} \Delta \sigma^3$. Therefore, we have to shift over to the monodromy matrix at the field limit defined as

$$T_f(\lambda) = U_0^{-N} T(\lambda) U_0^{-N}, \quad N \rightarrow \infty, \quad (35)$$

and for further construction introduce $V(\lambda, \mu) \equiv U_0(\lambda) \otimes U_0(\mu)$, $W(\lambda, \mu) = (U^j(\lambda) \otimes U^j(\mu))_{j \rightarrow \infty}$. We may check from the QYBE (28) that W satisfies the relation $R(\lambda - \mu)W(\lambda, \mu) = W(\mu, \lambda)R(\lambda - \mu)$, using which we can derive from QYBE (31), that the field monodromy matrix (35) also satisfies the QYBE

$$R_0(\lambda, \mu) T_f(\lambda) \otimes T_f(\mu) = T_f(\mu) \otimes T_f(\lambda) R_0(\lambda, \mu), \quad (36)$$

but with a transformed R -matrix:

$$R_0 = S(\mu, \lambda) R(\lambda - \mu) S(\lambda, \mu), \quad S(\lambda, \mu) = W^{-N} V^N, \quad N \rightarrow \infty, \quad (37)$$

where $R(\lambda - \mu)$ is the original rational quantum R -matrix (30) (see fadrev for similar details on 1d NLS model). Based on the above formulation, using the field operator products: $S_j^+ S_{j,x}^- = 2\frac{\alpha}{\Delta}$, $S_{j,x}^- S_j^+ = 0$, at $j \rightarrow \infty$, we can calculate explicitly the relevant objects needed for our field model. In particular, the central 2×2 block W_c for matrix W turns out to be

$$W_c(\lambda, \mu) = I + \Delta M(\lambda, \mu) \begin{pmatrix} (\lambda - \mu) & 0 \\ -2\alpha & -(\lambda - \mu) \end{pmatrix}, \quad (38)$$

with an intriguing factorization of its spectral dependence by a prefactor $M(\lambda, \mu) = 2\frac{(\lambda + \mu)}{\lambda^2 \mu^2}$, which is the key reason behind the success of the exact algebraic Bethe ansatz solution for our field model, inspite of the more complicated form of its Lax operator with nonlinear dependence on the spectral parameter and on the fields.

For constructing R_0 using definition (37), we have to find first the matrix $S(\lambda, \mu)$, taking proper limit of W^{-N} at $L \rightarrow \infty$ using (38). Through some algebraic manipulations, which are skipped here, we finally arrive at the field limit, to a simple form for R_0 matrix, expressed through its nontrivial elements as

$$\begin{aligned} R_{11}^{11} &= R_{22}^{22} = a(\lambda - \mu), \quad R_{21}^{12} = b(\lambda - \mu), \quad R_{22}^{11} = R_{11}^{22} = 0, \\ R_{12}^{21} &= b(\lambda - \mu) - \frac{\alpha^2}{\lambda - \mu} + \frac{\alpha^2 \pi}{M(\lambda, \mu)} \delta(\lambda - \mu), \end{aligned} \quad (39)$$

where $M(\lambda, \mu) = 2\frac{(\lambda + \mu)}{\lambda^2 \mu^2}$, $a(\lambda - \mu), b(\lambda - \mu)$ as in (30) and the $\delta(\lambda - \mu)$ term vanishes at $\lambda \neq \mu$. It is interesting to compare (39) with the original quantum R -matrix (30). Now from QYBE (36) relevant for the field models, we can derive using the R_0 matrix (39), the required CR between the operator elements of $T_f(\lambda)$. In particular, we get for our quantum field model the commutation relation

$$A_f(\lambda)B_f(\mu_a) = (f_a(\lambda - \mu_a) - \frac{\alpha^2 \pi \lambda^2 \mu^2}{2(\lambda + \mu_a)} \delta(\lambda - \mu_a)) B_f(\mu_a) A_f(\lambda), \quad (40)$$

where $f_a = \frac{\lambda - \mu_a - i\alpha}{\lambda - \mu_a}$. Note that at $\lambda \neq \mu_a$, the singular term with a prefactor bearing the imprint of the λ^2 dependence of our Lax matrix, vanishes and the relations coincide in parts with those of the known LLE model, though only formally, since the nature of the basic fields is completely different for these two models.

Using this result and the property of the vacuum state: $A_f|0\rangle = |0\rangle$, we obtain the exact EVP for

$$A_f(\lambda)|M\rangle = F_M|M\rangle, \quad \text{as } F_M = \prod_a^M f_a(\lambda - \mu_a), \quad A_f(\lambda)|0\rangle = |0\rangle \quad (41)$$

and hence for $\tau_f(\lambda)$, which yields finally the exact eigenvalues $c_n^{(M)}$ for conserved operators $C_n^{(M)}$ from the relation

$$\tau_f(\lambda)|M\rangle = \Lambda_M(\lambda)|M\rangle, \quad \ln \Lambda_M(\lambda) = \sum_n c_n^{(M)} \lambda^n, \quad (42)$$

all of which can be extracted systematically. Few lower ones from this infinite series take the explicit form

$$\begin{aligned} c_0^{(M)} &= \sum_{a=1}^M \rho_0(\mu_a), \quad \rho_0(\mu_a) = \frac{1}{2} \ln(1 + \frac{\alpha^2}{\mu_a^2}), \quad c_1^{(M)} = \sum_{a=1}^M \rho_1(\mu_a), \quad \rho_1(\mu_a) = \frac{2\alpha^2}{\mu_a(\alpha^2 + \mu_a^2)} \\ c_2^{(M)} &= \sum_{a=1}^M \rho_2(\mu_a), \quad \rho_2(\mu_a) = \alpha^2 \left[\frac{3\mu_a^2 + \alpha^2}{(\mu_a(\alpha^2 + \mu_a^2))^2} \right], \\ c_3^{(M)} &= \sum_{a=1}^M \rho_3(\mu_a), \quad \rho_3(\mu_a) = 2\alpha^2 \left[\frac{\mu_a^4 + 3\alpha^2 \mu_a^2 + \alpha^4}{3(\mu_a(\alpha^2 + \mu_a^2))^3} \right] \end{aligned} \quad (43)$$

etc. where $H = C_2$ is the Hamiltonian of our model. Therefore we obtain the exact energy spectrum as $E_M = c_2^{(M)}$, for the M-particle scattering state, which clearly differs from that of the known LLE model fadrev,qLLE. However, the overall spectrum of the conserved operators coincides in both these models due to the same quantum R -matrix involved in both these cases. Note, that due to the vacuum state property $A_f|0\rangle = |0\rangle$, the imprint of the Lax operators, which are widely different for the LLE and the present model, is lost at the field limit, leaving the R -matrix as the determining factor for the eigenvalues of the conserved operators.

It is interesting to compare the eigenvalues of the conserved operators (43) and their corresponding classical expressions (20,21). It is remarkable, that in spite of the highly nonlinear field interactions present in the Hamiltonian (21), the scattering spectrum shows no coupling between individual quasi-particles, mimicking a free-particle like scenario.

On the other hand, the bound-state or the quantum *soliton* state, which is obtained for the complex *string* solution for the particle momentum: $\mu_a^{(s)} = \mu_0 + i\frac{\alpha}{2}((M+1) - 2a)$, $a = 1, 2, \dots, M$ where μ_0 is the average particle momentum and α is the coupling constant, induces mutual interaction between the particles. Recall, that a bound-state becomes stable, when its energy is lower than the sum of the individual free-particle energies with the average momentum, which in turn is ensured by the negative values of the binding energy. More negative binding energy indicates more stable bound-states. The corresponding bound state energy spectrum

can be calculated for the present model for the $M > 1$ -particle bound-state, though it becomes rather cumbersome due to complicated expression of $c_2^{(M)}$ involving series sum of rational functions due to the rational dependence of the energy density $\rho_2(\mu_a)$ on μ_a . Though it is straightforward to extract the bound state energy, the resulting expression is lengthy, containing several terms involving polygamma functions and will not be reproduced here for the general case of $M > 2$. However To illustrate the situation and to demonstrate the intriguing stability condition for the bound state of the model we present only the simplest case for the energy of the 2-particle bound state given by the following expressions containing both positive E^+ and negative E^- contributions:

$$\begin{aligned}
E_2^{(s)} &= E^+ - E^- \\
E^+ &= \frac{1}{4}(16\alpha^3 + 8\alpha^2\mu_0^2 + \frac{48\alpha^2}{(\alpha^2 + 4\mu_0^2)^2} + \frac{(432\alpha^2 + 2176\alpha^7)}{(9\alpha^2 + 4\mu_0^2)^2} \\
&\quad + \frac{(12 + \alpha^4(32 + 1041\alpha^2))}{(9\alpha^2 + 4\mu_0^2)}) \tag{44}
\end{aligned}$$

$$E^- = \frac{1}{4}(66\alpha^4 + \frac{12}{(\alpha^2 + 4\mu_0^2)} + \frac{(320\alpha^6 + 4025\alpha^8)}{(9\alpha^2 + 4\mu_0^2)^2} + \frac{384\alpha^5}{(9\alpha^2 + 4\mu_0^2)}) \tag{45}$$

Note that when the 2-particle bound state energy $E_2^{(s)}$ for our quai-2d quantum field model becomes less than the sum of the energies: $2\rho_2(\mu_0)$ of two free particle scattering state, the bound state becomes stable due to the nontrivial value of the of the binding energy, which would be determined by the competing contributions of the positive (44) and negative (45) parts of the bound state eberg.

5. Concluding remarks and Outlook

Summarizing the saliant points of our construction we note, that since both the standard $(1+1)$ -dimensional LLE model and the present quasi $(2+1)$ -dimensional model in the quantum case are linked with the same R -matrix and the eigenvalues of the conserved operators are determined mainly by its c -number matrix elements, especially at the field limit, the eigenvalues coincide formally for the higher conserved operators in both the above models, although the energy spectrum corresponding to different Hamiltonians for these models are distinct. At the same time, though these two models are intimately related, the contents and the structure of these models are widely different, with different nature of their basic fields. The fields \mathbf{S} in the known LLE model behave like spin fields satisfying the $su(2)$ algebraic relations (6) and exhibits an important constraint $\mathbf{S}^2 = I$, as a Casimir operator. The associated Lax operator U_{lle} (7), generating shift along the x -direction, has only linear dependence on the spectral parameter and on the fields and satisfies as an ultralocal model the quantum Yang-Baxter equation (QYBE) along the x -axis. On the other hand, the basic fields of the present model satisfy commutation relations (27), which do not allow any constraint and the fields behave like three independent real scalar fields with nonstandard commutators, exhibiting unusual and significantly different nature of the fields. These novel CRs, involving x -derivative of the field are defined along the y axis, showing the quasi $(2+1)$ d character of the model, which is reflected also in the form of its conserved quantities (20). The related Lax operator $U(\lambda)$ (8), representing infinitesimal shift operator along the transverse direction y ,

has a nonlinear dependence on the spectral parameter as well as on the fields and contains x derivative of the field, showing higher scaling order and space-asymmetry of the model. In spite of these explicit unfavorable facts the quantum Lax operator of our model satisfies the crucial ultralocality condition and the QYBE with the rational quantum R -matrix, with y as the relevant direction, thanks to the unusual CRs of the fields.

The integrable model, proposed here, is important from several point of view. First, as a *new* integrable quantum field model satisfying the QYBE and exactly solvable by the algebraic Bethe ansatz, is important by its own right. Second, as a quantum field model built in quasi 2-dimensions, going beyond the standard construction of the existing 1-dimensional quantum integrable models and solved exactly by the Bethe ansatz, is a significant achievement. Third, as a quantum integrable model, constructed following the idea of higher order Lax operator, provides a nontrivial example of another new model in quasi 2-dimensions, needed for proving the conjecture and showing the universality of the approach proposed in Kundu [15]. Fourth, since the quantum LLE received renewed attention due to its link with the string theory, following the ADS/CFT correspondence, the quantum field model proposed here, due to its close proximity with the quantum LLE, could also be interesting from other angles.

The idea followed here should show the path in constructing a novel class of higher-dimensional field and lattice models both at the classical and the quantum level and should help in discovering new type of algebraic relations, like those found here.

6. References

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